Pergamon

Int. J. Heat Mass Transfer. Vol. 38, No. 18, pp. 3393–3399, 1995

0017–9310/95 \$9.50+0.00

0017–9310(95)00110–7

# Heat conduction in a semi-infinite solid subject to steady and non-steady periodic-type surface heat fluxes

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(*Received* 23 *August* 1993 *and in final form* 16 *February* 1995)

**Abstract**—An analytical solution for the temperature and heat flux distribution in the case of a semi-infinite solid of constant properties is investigated. The solutions are presented for time-dependent, surface heat fluxes of the forms: (i) $\dot{Q}_1(t) = \dot{Q}_0(1+a\cos\omega t)$; and (ii) $\dot{Q}_2(t) = \dot{Q}_0(1+bt\cos\omega t)$, where $a$ and $b$ are controlling factors of the periodic oscillations about the constant surface heat flux $\dot{Q}_0$. The dimensionless (or reduced) temperature and heat flux solutions are presented in terms of decompositions $C_\Gamma$ and $S_\Gamma$ of the generalized representation of the incomplete Gamma function. It is demonstrated that the present analysis covers the limiting case for large times which is discussed in several textbooks, for the case of steady periodic-type surface heat fluxes. In addition, an illustrative example problem on heating of malignant tissues, making use of transient and long-time solutions, is also presented.

## 1. INTRODUCTION

There are several heat conduction problems which can be approximated by a semi-infinite solid of constant properties. These heat transfer problems might be in terms of finite geometries where, for short times, the heating or cooling effects at the surface have not yet been felt very far into the material; for example, a step change in the surface temperature of a thick sheet of metal at small enough times, so that temperature of the center is still at its initial value. Another problem of engineering is the penetration of the daily and annual temperature cycles into the earth's surface. The earth can be considered as a semi-infinite solid since its radius is so much larger than the depth to which the thermal fluctuations penetrate.

It should be noted that these problems are typically solved by transform techniques, and solutions of several such problems are discussed in the heat transfer literature [1, 2]. However, a recent interest in the laser-induced processing of materials has further motivated engineers and scientists to explore some additional solutions of the semi-infinite solid which are not covered in the literature. Recently, Zubair and Chaudhry [3] have presented some closed-form temperature and heat flux solutions of a semi-infinite solid subject to time-dependent surface heat fluxes. They have discussed the solutions for power-type, exponential-type and pulse-type surface heat fluxes. There are many engineering problems in which periodic boundary conditions must be considered. These are typically encountered in: (i) the study of fluctuations in temperature of the earth's crust due to periodic heating by the sun; (ii) various experimental arrangements for the determination of thermal diffusivity of materials; (iii) the calculation of the periodic temperatures (and thus of periodic stresses) in the cylinder walls of internal combustion engines; and (iv) the theory of automatic temperature control systems. The objective of this paper is to present and discuss closed-form solutions for the following important cases of time-dependent boundary conditions:

$$\dot{Q}_1(t) = \dot{Q}_0(1+a\cos\omega t) \tag{1}$$

$$\dot{Q}_2(t) = \dot{Q}_0(1+bt\cos\omega t) \tag{2}$$

where $a$ and $b$ are controlling factors of the periodic oscillations about the constant surface heat flux, $\dot{Q}_0$. We note that equation (1) represents the steady, periodic heat flux oscillations, while for the starting periodic-type oscillations, equation (2) may be used.

## 2. MATHEMATICAL FORMULATION

We consider a semi-infinite homogeneous and isotropic body of initial temperature $T(x,0) = g(x)$ whose surface temperature is subjected to a heat flux given by $-k[\partial T(0,t)/\partial x] = f(t)$. The solution to this problem is described in Appendix A by equation (A7) as

$$T(x,t) = g(x) + \frac{\alpha^{1/2}}{k\pi^{1/2}}\int_0^t f(\tau) \times \exp\left[\frac{-x^2}{4\alpha(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{1/2}}. \tag{3}$$

The substitution

**NOMENCLATURE**

| | |
|---|---|
| $a$ | controlling factor for steady periodic oscillations |
| $b$ | controlling factor for non-steady periodic oscillations |
| $C_p$ | specific heat at constant pressure [kJ kg$^{-1}$ K$^{-1}$] |
| $C_\Gamma$ | decomposition function |
| $Fo$ | Fourier number, $Fo = \alpha t/x^2$ |
| $k$ | thermal conductivity [W m$^{-1}$ K$^{-1}$] |
| $q''_x$ | heat flux in the $x$ direction [W m$^{-2}$] |
| $Q$ | reduced (or dimensionless) heat flux $\{2\pi^{1/2}[q''_x + k(\mathrm{d}g/\mathrm{d}x)]/\dot{Q}_0\}$ |
| $\dot{Q}$ | surface-heat flux [W m$^{-2}$] |
| $S_\Gamma$ | decomposition function |
| $t$ | time [s] |
| $T$ | temperature [K] |
| $x$ | spatial variable [m]. |

Greek symbols

| | |
|---|---|
| $\alpha$ | thermal diffusivity, $\alpha = k/\rho C_p$ [m$^2$ s$^{-1}$] |
| $\Gamma$ | Gamma function |
| $\theta$ | reduced (or dimensionless) temperature $[2\pi^{1/2}k\{T(x,t)-g(x)\}/x\dot{Q}_0]$ |
| $\rho$ | density [kg m$^{-3}$] |
| $\tau$ | reduced or dimensionless time, $\tau = \omega t$ |
| $\tau_o$ | dimensionless controlling factor, $\tau_o = bt$ |
| $\omega$ | frequency [s$^{-1}$]. |

Subscripts

| | |
|---|---|
| i | initial |
| o | constant |
| 1 | steady, periodic surface heat flux, $\dot{Q}_1 = \dot{Q}_0(1+a\cos\omega t)$ |
| 11 | constant surface heat flux |
| 12 | long-time solution |
| 2 | non-steady periodic surface heat flux, $\dot{Q}_2 = \dot{Q}_0(1+bt\cos\omega t)$. |

$$\phi = \frac{x^2}{4\alpha(t-\tau)} \quad \text{and hence} \quad \left(\sqrt{\frac{x^2}{4\alpha}}\right)\frac{\mathrm{d}\phi}{\phi^{3/2}} = \frac{\mathrm{d}\tau}{(t-\tau)^{1/2}}$$

allows us to reduce equation (3) to the form

$$T(x,t) = g(x) + \frac{\alpha^{1/2}}{k\pi^{1/2}}(x^2/4\alpha)^{1/2} \times \int_{x^2/4\alpha t}^{\infty} f(t - x^2/4\alpha\phi)\exp(-\phi)\frac{\mathrm{d}\phi}{\phi^{3/2}}. \quad (4)$$

It should be noted that the solution given by equation (4) satisfies the differential equation and the initial and boundary conditions.

### 2.1. *The steady-periodic surface heat flux*

We note that a steady, periodic-type surface heat flux variation given by equation (1), when substituted in equation (4), results in

$$T_1(x,t) = g(x) + \frac{x\dot{Q}_0}{2k\pi^{1/2}}\left[\int_{x^2/4\alpha t}^{\infty}\exp(-\phi)\frac{\mathrm{d}\phi}{\phi^{3/2}} + a\cos(\omega t)\int_{x^2/4\alpha t}^{\infty}\exp(-\phi)\cos\left(\frac{\omega x^2}{4\alpha\phi}\right)\frac{\mathrm{d}\phi}{\phi^{3/2}} + a\sin(\omega t)\int_{x^2/4\alpha t}^{\infty}\exp(-\phi)\sin\left(\frac{\omega x^2}{4\alpha\phi}\right)\frac{\mathrm{d}\phi}{\phi^{3/2}}\right]. \quad (5)$$

The solution of the above equation can be written in terms of the functions $C_\Gamma(\alpha, x; b)$, defined by equations (B1) and (B2), respectively. This gives

$$T_1(x,t) = g(x) + \frac{x\dot{Q}_0}{2k\pi^{1/2}}[\Gamma(-1/2, x^2/4\alpha t) + a\cos(\omega t) \times C_\Gamma(-1/2, x^2/4\alpha t; \omega x^2/4\alpha) + a\sin(\omega t)S_\Gamma(-1/2, x^2/4\alpha t; \omega x^2/4\alpha)]. \quad (6)$$

Simplifying, we find in terms of dimensionless variables that

$$\theta_1 = \Gamma(-1/2, 1/4Fo) + a[\cos(\tau)C_\Gamma(-1/2, 1/4Fo; \tau/4Fo) + \sin(\tau)S_\Gamma(-1/2, 1/4Fo; \tau/4Fo)] \quad (7)$$

where $\theta_1$, $Fo$ and $\tau$ are defined in the Nomenclature.

On using the differentiation formula for $C_\Gamma(\alpha, x; b)$ and $S_\Gamma(\alpha, x; b)$ given by equations (B13) and (B14), respectively, the heat flux at any $x$ is

$$q''_{x,1} = -k\frac{\partial T_1}{\partial x} = -k\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\dot{Q}_0}{2\pi^{1/2}}\{2\Gamma(1/2, x^2/4\alpha t) + 2a(x^2/4\alpha t)\exp(-x^2/4\alpha t) + a\cos(\omega t)[2C_\Gamma(1/2, x^2/4\alpha t; \omega x^2/4\alpha) - 2(x^2/4\alpha t)^{-1/2}\exp(-x^2/4\alpha t)\cos(\omega t)] + a\sin(\omega t)[2S_\Gamma(1/2, x^2/4\alpha t; \omega x^2/4\alpha) - 2(x^2/4\alpha t)^{-1/2}\exp(-x^2/4\alpha t)\sin(\omega t)]\}. \quad (8)$$

Simplifying by using equations (B4) and (B5), we find in terms of dimensionless variables that

$$Q_1 = \Gamma(1/2, 1/4Fo) + a[\cos(\tau)C_\Gamma(1/2, 1/4Fo; \tau/4Fo) + \sin(\tau)S_\Gamma(1/2, 1/4Fo; \tau/4Fo)]. \quad (9)$$

We note that $a = 0$ reduces to the case of constant surface heat flux. On substituting this value of $a$ in equations (7) and (9), we obtain

$$\theta_{11} = \Gamma(-1/2, 1/4Fo) = 2[(1/4Fo)^{-1/2}\exp(-1/4Fo) - \pi^{1/2}\operatorname{erf}c(1/2\sqrt{Fo})] \quad (10)$$

$$Q_{11} = \Gamma(1/2, 1/4Fo) = \pi^{1/2}\operatorname{erf}c(1/2\sqrt{Fo}) \quad (11)$$

which are the same temperature and heat flux solutions as those reported by Carslaw and Jaeger [1].

It should be noted that for large values of time $t$, $\tau \to \infty$ and $1/Fo \to 0$. On substituting these values in equations (7) and (9), we find that

$$\theta_{12} = -2[(\sqrt{\pi}) - (1/4Fo)^{-1/2}\exp(-1/4Fo)] + a[\cos(\tau)C_\Gamma(-1/2, 0; \chi) + \sin(\tau)S_\Gamma(-1/2, 0; \chi)] \quad (12)$$

$$Q_{12} = (\sqrt{\pi}) + a[\cos(\tau)C_\Gamma(1/2, 0; \chi) + \sin(\tau)S_\Gamma(1/2, 0; \chi)] \quad (13)$$

where $\chi = \omega x^2/4\alpha$.

The above solutions can be simplified further by using equations (B7)–(B10), to give

$$\theta_{12} = -2[(\sqrt{\pi}) - (1/4Fo)^{-1/2}\exp(-1/4Fo)] + \frac{a\sqrt{\pi}}{\sqrt{\chi}}\exp(-\sqrt{2\chi})\cos[(\sqrt{2\chi}) + \pi/4 - \tau] \quad (14)$$

$$Q_{12} = \pi^{1/2}\{1 + a\exp(-\sqrt{2\chi})[\cos(\tau - \sqrt{2\chi})]\}. \quad (15)$$

The graphical representations of equations (7) and (9) are shown in Figs. 1 and 2, respectively. In these figures reduced temperature and heat fluxes are plotted in terms of reduced time parameters $\tau$ and $Fo$, for the case when the controlling factor $a$ of the steady-periodic oscillations is 0.5. As expected, we note that the amplitude of temperature and flux oscillations increases with $Fo$, however, for large values of $\tau$ these oscillations die away. On comparing reduced temperature plots with reduced heat flux plots, we note that the temperature and heat flux variations are out of phase with each other.

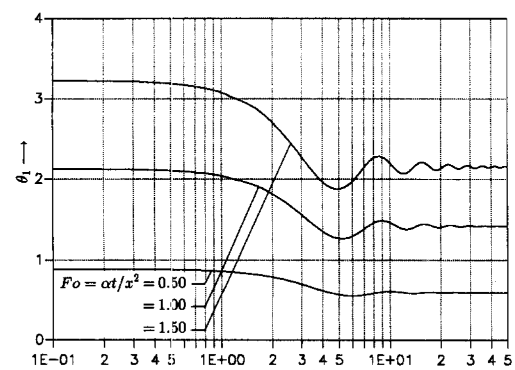


Fig. 1. Dimensionless temperature profiles as a function of $\tau$ due to a steady-periodic heat flux, for a controlling factor $a = 0.5$.

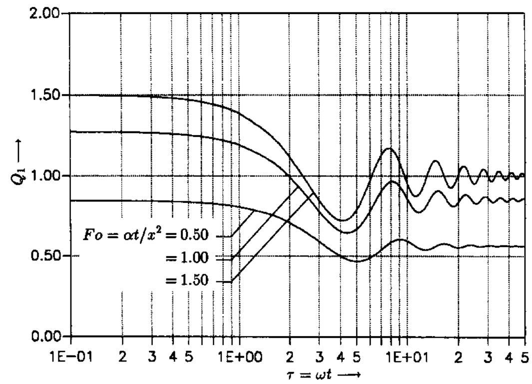


Fig. 2. Dimensionless heat flux profiles as a function of $\tau$ due to a steady-periodic heat flux, for a controlling factor $a = 0.5$.

It is interesting to compare long-time solutions with the transient solutions for reduced temperature and heat fluxes. Figure 3 shows the comparison of reduced temperature plots, while Fig. 4 shows the heat flux plots for the case when the controlling factor $a$ of the steady-periodic oscillations is 0.5 and $Fo = 10.0$. It

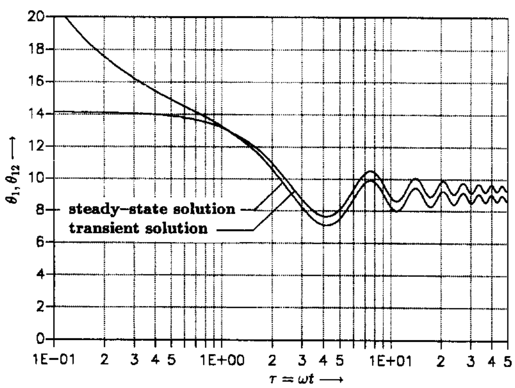


Fig. 3. Dimensionless temperature profiles as a function of $\tau$ due to a steady-periodic heat flux, for a controlling factor $a = 0.5$ and $Fo = 10.00$.

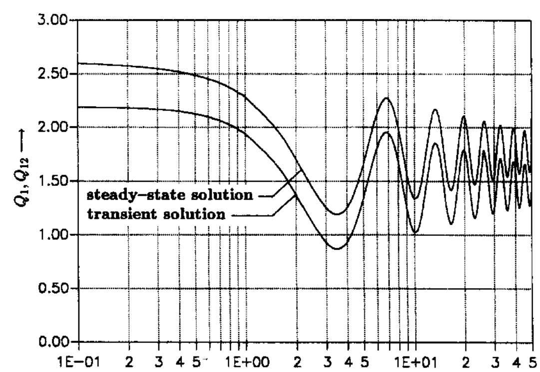


Fig. 4. Dimensionless heat flux profiles as a function of $\tau$ due to a steady-periodic heat flux, for a controlling factor $a = 0.5$ and $Fo = 10.00$.

can be seen from Fig. 3 that for low values of $\tau$, there is a significant difference between the steady-state and transient temperatures, whereas for large values of $\tau$, they approach the constant value. On the other hand, reduced heat flux plots of Fig. 4 show a somewhat constant difference between the steady-state and transient solutions.

### 2.2. *The starting-periodic, surface heat flux*

The starting periodic-type, surface heat flux given by equation (2), when substituted into equation (4), results in

$$T_2(x,t) = g(x) + \frac{x\dot{Q}_0}{2k\pi^{1/2}}\left[\int_{x^2/4\alpha t}^{\infty} \exp(-\phi)\frac{\mathrm{d}\phi}{\phi^{3/2}}\right.$$

$$+ (bt)\cos(\omega t)\int_{x^2/4\alpha t}^{\infty} \exp(-\phi)\cos\left(\frac{\omega x^2}{4\alpha\phi}\right)\frac{\mathrm{d}\phi}{\phi^{3/2}}$$

$$\times (bt)\sin(\omega t)\int_{x^2/4\alpha t}^{\infty} \exp(-\phi)\sin\left(\frac{\omega x^2}{4\alpha\phi}\right)\frac{\mathrm{d}\phi}{\phi^{3/2}}$$

$$+\left(\frac{bx^2}{4\alpha}\right)\cos(\omega t)\int_{x^2/4\alpha t}^{\infty} \exp(-\phi)$$

$$\times \cos\left(\frac{\omega x^2}{4\alpha\phi}\right)\frac{\mathrm{d}\phi}{\phi^{5/2}} + \left(\frac{bx^2}{4\alpha\phi}\right)\sin(\omega t)$$

$$\left.\times \int_{x^2/4\alpha t}^{\infty} \exp(-\phi)\sin\left(\frac{\omega x^2}{4\alpha\phi}\right)\frac{\mathrm{d}\phi}{\phi^{5/2}}\right] \tag{16}$$

which can be simplified in terms of dimensionless variables by using equations (B1) and (B2), to give

$$\theta_2 = \Gamma(-1/2, 1/4Fo)$$
$$+\tau_o[\cos(\tau)C_\Gamma(-1/2, 1/4Fo;\tau/4Fo)$$
$$+\sin(\tau)S_\Gamma(-1/2, 1/4Fo;\tau/4Fo)]$$
$$+(\tau_o/4Fo)[\cos(\tau)C_\Gamma(-3/2, 1/4Fo;\tau/4Fo)$$
$$+\sin(\tau)S_\Gamma(-3/2, 1/4Fo;\tau/4Fo)] \tag{17}$$

where $\tau_o$, $\theta_2$ and other dimensionless variables are defined in the Nomenclature.

The heat flux at any $x$ is calculated by differentiating equation (17) and using equations (B4), (B5), (B13) and (B14), to give

$$Q_2 = \Gamma(1/2, 1/4Fo) + \tau_o[\cos(\tau)C_\Gamma(1/2, 1/4Fo;\tau/4Fo)$$
$$+\sin(\tau)S_\Gamma(1/2, 1/4Fo;\tau/4Fo)]$$
$$+(\tau_o/4Fo)[\cos(\tau)C_\Gamma(-1/2, 1/4Fo;\tau/4Fo)$$
$$+\sin(\tau)S_\Gamma(-1/2, 1/4Fo;\tau/4Fo)]. \tag{18}$$

The reduced temperature and heat flux plots for a non-steady periodic surface heat flux vs the reduced time $\tau$ and controlling parameter ($\tau_o = bt$) are shown in Figs. 5 and 6, respectively. All these curves are drawn for $Fo = 1.00$. We note that, as expected, the amplitude of temperature and heat flux oscillations in these figures is a strong function of the reduced controlling factor. It should, however, be noted that for large values of $\tau$, the reduced temperature and heat flux oscillations die away regardless of the controlling parameter $\tau_o$.

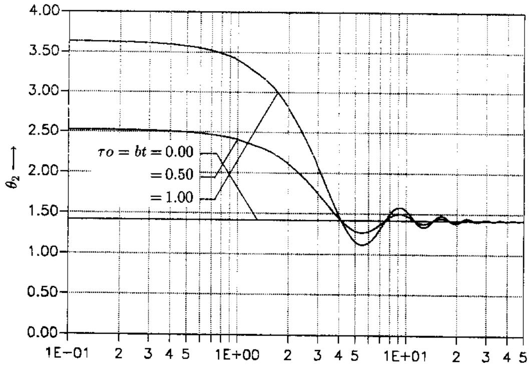


Fig. 5. Dimensionless temperature profiles as a function of $\tau$ due to a starting-periodic surface heat flux, for $Fo = 1.00$; the effect of controlling factor $\tau_0$.

## 3. ILLUSTRATIVE EXAMPLE

Heating of malignant tissues for therapeutic purposes using specially designed heating–cooling devices has recently become very attractive [4, 5]. The treatment, known as hyperthermia, involves heating of tumorous tissues in the temperature range 42–46°C for a specified period of time; usually 30–60 min is recommended. It should be noted that the temperature values within this range are not directly harmful to normal cells, while it is expected that cancerous cells are destroyed as they are more sensitive to high temperatures.

During a therapeutic treatment a specially designed heating–cooling device is attached to the back of a patient such that the heat flux provided to the patient

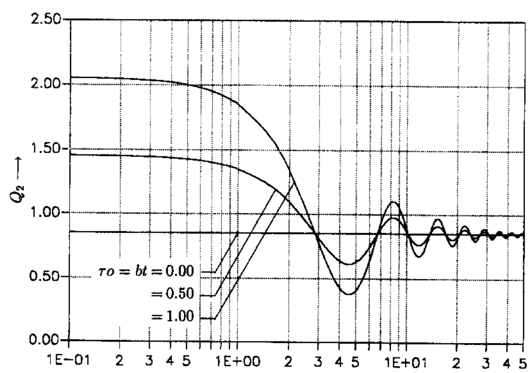


Fig. 6. Dimensionless heat flux profiles as a function of $\tau$ due to a starting-periodic surface heat flux, for $Fo = 1.00$; the effect of controlling factor $\tau_0$.

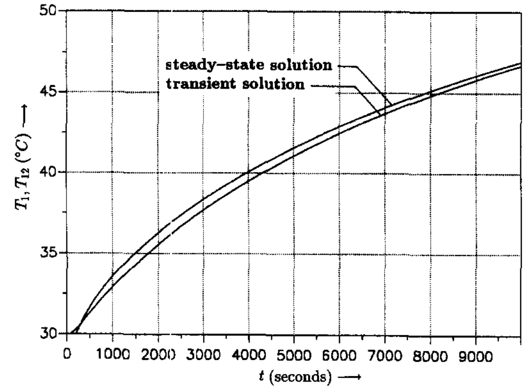


Fig. 7. Temperature profiles at a depth of 10 mm from the surface of the skin; for $\omega = 1\times10^{-4}$ rad s$^{-1}$, $T_i = 30$°C, $a = 0.5$ and $\dot{Q}_0 = 200$ W m$^{-2}$.

is a periodic function of time: $\dot{Q}_{back} = \dot{Q}_0(1+a\cos\omega t)$, where $a$ is the controlling factor of the periodic oscillations about the constant surface heat flux $\dot{Q}_0$. Modeling the patient's body as a flat wall of large (semi-infinite) thickness and known properties ($\rho = 1000$ kg m$^{-3}$, $C = 4180$ J kg$^{-1}$ K$^{-1}$, $k = 0.5016$ W m$^{-1}$ K$^{-1}$), it is desired to study the temperature and the heat flux distributions at a depth of 10 mm from the surface of the skin using both the transient and long-time solutions.

The results discussed below are based on the frequency of periodic oscillation $\omega = 1\times10^{-4}$ rad s$^{-1}$, the initial temperature of the patient's body $T_i = 30$°C, the controlling factor $a = 0.5$ and the constant surface heat flux $\dot{Q}_0 = 200$ W m$^{-2}$. On using equations (6), (8), (14) and (15), the temperature and heat flux distributions 10 mm from the surface of the skin are shown in Figs. 7 and 8, respectively. In these figures, the results are presented as a function of time from the start of the therapeutic treatment. It can be seen from the figures that there is a discrepancy between the transient and long-time results, particularly for heat flux variations. Figure 7 shows that the difference between the long-time and transient temperatures is somewhat negligible; however, the long-term solution predicts more temperature values as compared with transient results. For example, in the critical temperature range (42–46°C), the long-time solution shows that the desired temperature will be reached in about 1.39 h compared with 1.50 h predicted by the transient solution. In addition, the transient results indicate that the heater can maintain the required temperature for about 1.0 h. On the other hand, Fig. 8 shows a considerable difference between the heat fluxes at the start-up of the system. As expected, the heat flux predicted by the transient solution is initially zero, then increases steadily and reaches an asymptotic value, while the long-time solution predicts a more or less constant heat flux with time.

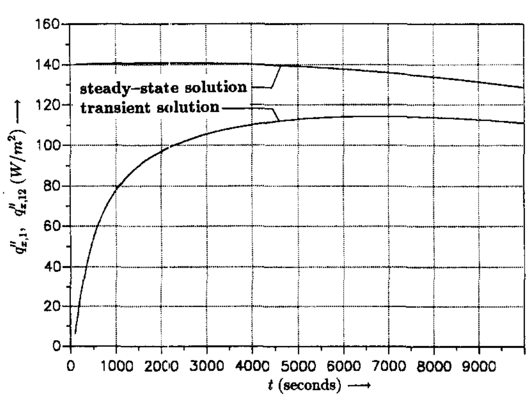


Fig. 8. Heat flux profiles at a depth of 10 mm from the surface of the skin; for $\omega = 1\times10^{-4}$ rad s$^{-1}$, $T_i = 30$°C, $a = 0.5$ and $\dot{Q}_0 = 200$ W m$^{-2}$.

## 4. CONCLUDING REMARKS

The closed-form solutions of temperature and heat flux distributions due to time-dependent surface heat fluxes are discussed for steady and non-steady, periodic-type surface heat fluxes in a semi-infinite, homogeneous and isotropic solid. The reduced temperature and heat flux solutions are presented as a function of reduced time parameters $\tau$ and $Fo$. It is demonstrated that the long-time solution for the steady-periodic case reduces to the classical steady-state solution presented in several textbooks. The limits for the long-time solutions are discussed in terms of $Fo$. It is also shown that when the controlling factors $a$ and $b$ of the periodic oscillations are equal to zero, the reduced temperature and heat flux solutions of the steady and non-steady periodic surface heat fluxes are reduced to the classical solution for the case of constant surface heat flux boundary condition.

Furthermore, a numerical example on heating of malignant tissues is used to demonstrate a difference between the long-time and transient solutions. The results show that the time to reach the critical temperature (42–46°C) for the therapeutic treatment is strongly dependent on the chosen solution. For example, the long-time solution shows that the desired temperature will be reached in about 1.39 h compared with 1.50 h predicted by the transient solution when the frequency of periodic oscillation $\omega = 1\times10^{-4}$ rad s$^{-1}$, the initial temperature $T_i = 30$°C, the controlling factor $a = 0.5$, and the constant surface heat flux $\dot{Q}_0 = 200$ W m$^{-2}$.

*Acknowledgement*—The authors acknowledge the support provided by the King Fahd University of Petroleum and Minerals for this research project.

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## APPENDIX A

In the following analysis, we discuss the use of Duhamel's method [1, 2] for the solution of heat conduction problems subject to time-dependent heat flux boundary conditions. The governing equations are

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \tag{A1a}$$

with

$$T(x,0) = g(x), \quad g''(x) = 0 \tag{A1b}$$

$$-k\frac{\partial T(t)}{\partial x} = f(t). \tag{A1c}$$

Consider the following auxiliary problem in which $\Phi_\tau(x,t)$ is the solution of problem (A1) on the assumption that $\Phi_\tau(t)$ is independent of $t$, i.e.

$$\frac{\partial^2 \Phi_\tau(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Phi_\tau(x,t)}{\partial t} \tag{A2a}$$

with

$$\Phi_\tau(x,0) = g(x) \tag{A2b}$$

$$-\frac{k\,\partial\Phi_\tau(t)}{\partial x} = f(\tau). \tag{A2c}$$

The Laplace transform of equations (A2a)–(A2c) with respect to $t$ gives

$$\frac{\partial^2 \bar{\Phi}_\tau(x,s)}{\partial x^2} = \frac{s}{\alpha}\bar{\Phi}_\tau(x,s) - g(x)/s \tag{A3a}$$

$$\bar{\Phi}_\tau(x,0) = g(x)/s \tag{A3b}$$

$$-k\frac{\partial\bar{\Phi}_\tau(0,s)}{\partial x} = f(\tau)/s. \tag{A3c}$$

The solution of system (A3) can be written as

$$\bar{\Phi}_\tau(x,s) = \frac{\alpha^{1/2} f(\tau)\,e^{-(\sqrt{s/\alpha})x}}{ks^{3/2}} + g(x)/s. \tag{A4}$$

Taking the inverse Laplace transform, we obtain

$$\Phi_\tau(x,t) = \frac{\alpha^{1/2} f(\tau)}{k}\left[\frac{2}{\pi^{1/2}}t^{1/2}\,e^{-x^2/4\alpha t} - \frac{x}{\alpha^{1/2}}\operatorname{erf}c\left(\frac{x}{\sqrt{4\alpha t}}\right)\right] + g(x). \tag{A5}$$

According to Duhamel's theorem [1–3], the solution to the system (A1) is

$$T(x,t) = \frac{\partial}{\partial t}\left[\int_0^t \Phi_\tau(x,t-\tau)\,d\tau\right]. \tag{A6}$$

Using Leibnitz's rule of differentiation, we get

$$T(x,t) = \Phi_\tau(x,0) + \int_0^t \frac{\partial}{\partial t}[\Phi_\tau(x,t-\tau)\,d\tau]$$

or

$$= g(x) + \frac{\alpha^{1/2}}{k\pi^{1/2}}\int_0^t f(\tau)\exp\left[\frac{-x^2}{4\alpha(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{1/2}} \tag{A7}$$

where

$$\frac{d}{dt}\left[\operatorname{erf}c\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] = \frac{x/2}{\sqrt{\pi\alpha}}\frac{\exp(-x^2/4\alpha t)}{t^{3/2}}. \tag{A8}$$

## APPENDIX B

It should be noted that the integrals occurring in equations (5) and (16) can be represented as [6]

$$C_\Gamma(\alpha,x;b) = \int_x^\infty t^{\alpha-1}\,e^{-t}\cos(b/t)\,dt \tag{B1}$$

and

$$S_\Gamma(\alpha,x;b) = \int_x^\infty t^{\alpha-1}\,e^{-t}\sin(b/t)\,dt. \tag{B2}$$

These functions belong to the family of the Weyl fractional integrals [7], and satisfy the following formulae [8]:

$$C_\Gamma(\alpha,x;\omega) - iS_\Gamma(\alpha,x;\omega) = \Gamma(\alpha,x;i\omega) \tag{B3}$$

$$C_\Gamma(\alpha+1,x;\omega) = \alpha C_\Gamma(\alpha,x;\omega) + \omega S_\Gamma(\alpha-1,x;\omega) + x^\alpha e^{-x}\cos(\omega/x) \tag{B4}$$

$$S_\Gamma(\alpha+1,x;\omega) = \alpha S_\Gamma(\alpha,x;\omega) - \omega C_\Gamma(\alpha-1,x;\omega) + x^\alpha e^{-x}\sin(\omega/x). \tag{B5}$$

A tabular and graphical representation of the functions $C_\Gamma$ and $S_\Gamma$ is given in ref. [8]. We note that for $b = 0$, equation (B1) reduces to

$$C_\Gamma(\alpha,x;0) = \Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}\,e^{-t}\,dt. \tag{B6}$$

It should be noted that for $x = 0$ and $\alpha = \frac{1}{2}\pm n$, $n = 0, 1, 2, 3, \ldots$, the real and imaginary parts in equation (B3) can be simplified in terms of trigonometric functions [8]. In particular, we have

$$C_\Gamma(1/2,0;\omega) = \pi^{1/2}\exp(-\sqrt{2\omega})\cos(\sqrt{2\omega}) \tag{B7}$$

$$S_\Gamma(1/2,0;\omega) = \pi^{1/2}\exp(-\sqrt{2\omega})\sin(\sqrt{2\omega}) \tag{B8}$$

$$C_\Gamma(-1/2,0;\omega) = \frac{\pi^{1/2}}{\sqrt{\omega}}\exp(-\sqrt{2\omega})\cos[(\sqrt{2\omega})+\pi/4], \tag{B9}$$

$$S_\Gamma(-1/2,0;\omega) = \frac{\pi^{1/2}}{\sqrt{\omega}}\exp(-\sqrt{2\omega})\sin[(\sqrt{2\omega})+\pi/4] \tag{B10}$$

$$C_\Gamma(-3/2,0;\omega) = \frac{\pi^{1/2}}{\sqrt{\omega}}\exp(-\sqrt{2\omega}) \times\left[\frac{\sin\sqrt{2\omega}}{\sqrt{\omega}} + \frac{\sin[(\sqrt{2\omega})+\pi/4]}{2\omega}\right] \tag{B11}$$

$$S_\Gamma(-3/2, 0; \omega) = \frac{-\pi^{1/2}}{\sqrt{\omega}} \exp(-\sqrt{2\omega}) \times \left[ \frac{\cos\sqrt{2\omega}}{\sqrt{\omega}} + \frac{\cos[(\sqrt{2\omega}) + \pi/4]}{2\omega} \right] \quad \text{(B12)}$$

and the differentiation formulae for $C_\Gamma(\alpha, x; bx)$ and $S_\Gamma(\alpha, x; bx)$ with respect to $x$ are given by

$$\frac{\partial}{\partial x}[C_\Gamma(\alpha, x; bx)] = -[bS_\Gamma(\alpha-1, x; bx) + x^{\alpha-1}\,\mathrm{e}^{-x}\cos(b)] \quad \text{(B13)}$$

and

$$\frac{\partial}{\partial x}[S_\Gamma(\alpha, x; bx)] = [bC_\Gamma(\alpha-1, x; bx) - x^{\alpha-1}\,\mathrm{e}^{-x}\sin(b)]. \quad \text{(B14)}$$